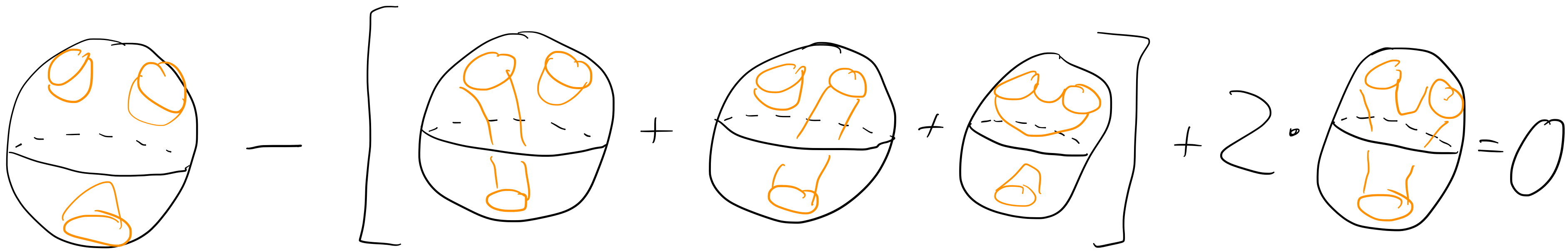
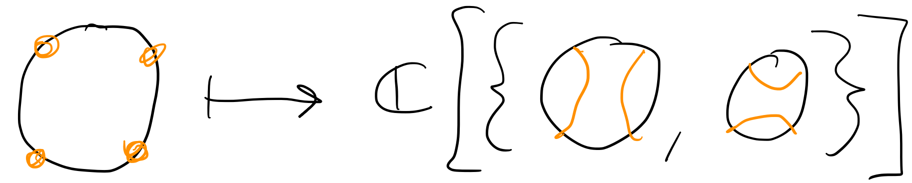
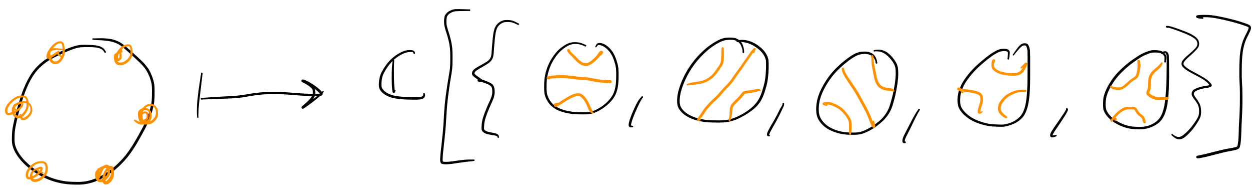
Higher Temperley-Lieb categories, Orthogonal Polynomials, and (3+ε)-dimensional TQFTs



Temperley-Lieb 2-category review

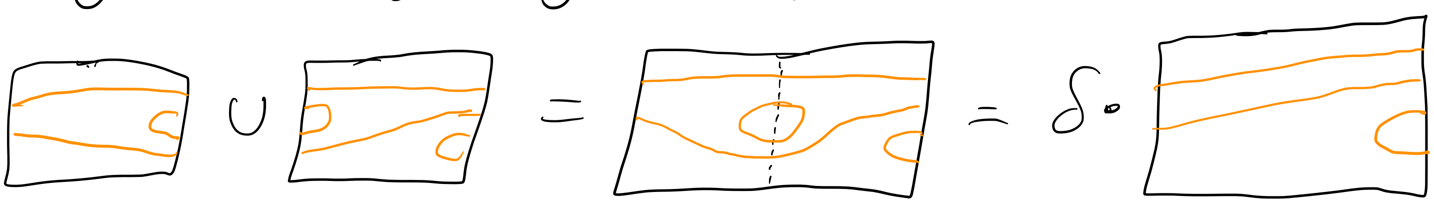
Disk with ∂-condition $c \rightsquigarrow$ vector space $A(D^2; c)$



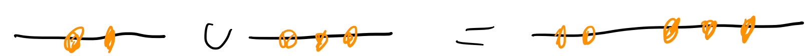


$$\dim(A(D^2; c)) = \begin{cases} 0, & |c| \text{ odd} \\ \text{catalan}(|c|), & |c| \text{ even} \end{cases}$$

Can glue disks together:



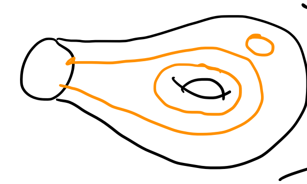
Can glue ∂-cond together:



$\Rightarrow$ We have a 2-category (linear, strict pivotal)

$$\{n\text{-categories}\} \longleftrightarrow \{(n+\varepsilon)\text{-dimensional TQFTs}\}$$

Some components of the TL TQFT:

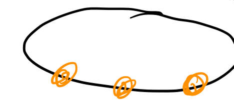
(2) $(Y^2, c) \longmapsto A(Y; c) = \mathbb{C}\Big[$  $\Big] / \sim$

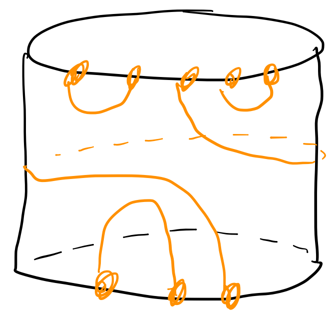
- c: ∂-condition
- $A(Y;c)$: predual Hilbert space
- $\sim$: local relations (isotopy rel ∂, $\bigcirc = \delta \cdot \emptyset$)

(2-dim'l skein module)

① $X^1 \longmapsto A(X) =$

(1-manifold, e.g. I or S^1) (cylinder category)

- objects: $\{$  $\}$ ← pts on X
- $mor(a \to b) = A(X \times I; a, b)$



- composition: stacking cylinders

③ Path integral

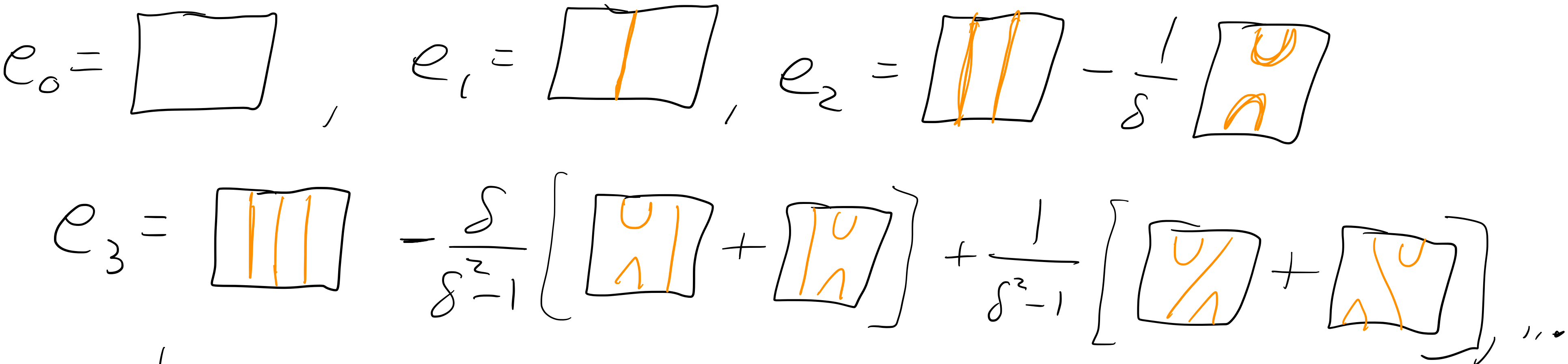
$$Z(M^3) : A(\partial M) \longrightarrow \mathbb{C}$$

(only defined when $M =$ 0-handles $\cup$ 1-handles)

A

Orthogonal polynomials from the TL TQFT

1-cat $A(I)$ ⇝ minimal idempotents:

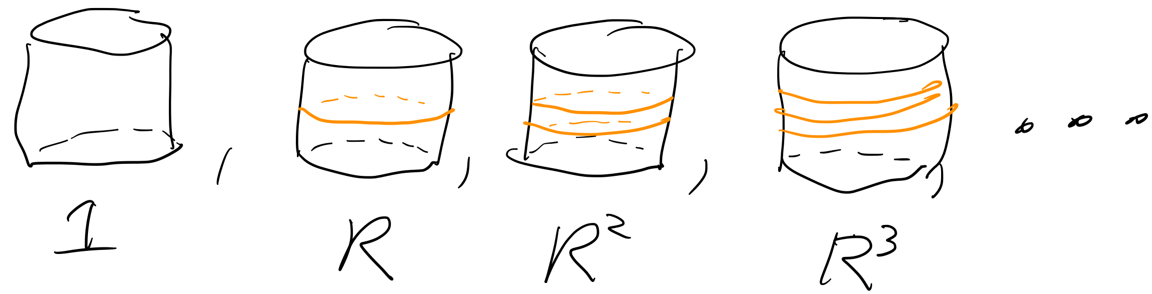


etc.

("Jones–Wenzl idempotents")

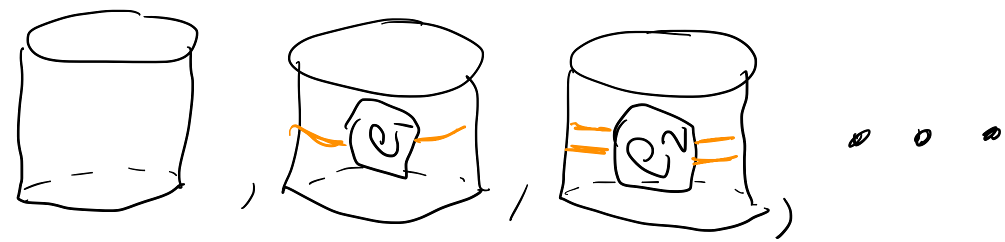
1-cat $A(S^1) \supset \mathrm{mor}(\emptyset \to \emptyset)$

- "topological basis":



polynomial algebra $\mathbb{C}[R]$

- "algebraic basis":



- trace/inner product on morphisms of $A(S^1)$

$$\mathrm{tr}(X) = Z(S^1 \times I \times I)(\mathrm{cl}(X))$$

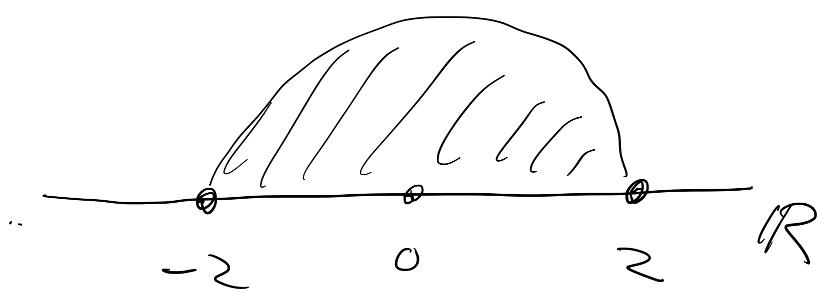
$$\langle X, Y \rangle = \mathrm{tr}(\bar{X} \cdot Y)$$

path integral — closure of X $\in A(S^1 \times S^1)$

★ algebraic basis is orthonormal w.r.t. I.P.,
so we get a family of **orthogonal polynomials** ← Chebyshev polynomials

$$\langle cl(e_i), cl(e_j) \rangle = \delta_{ij}$$

- trace ⇝ measure on $\mathbb{R}$
  moments of measure: $\operatorname{tr}(R^k) = \begin{cases} 0, & k \text{ odd} \\ \operatorname{cat}(k), & k \text{ even} \end{cases}$ ← Catalan number

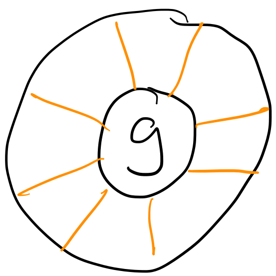


$$d\mu = c \cdot \sqrt{1 - \left(\frac{r}{2}\right)^2}\, dr$$

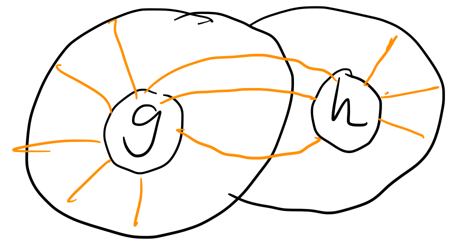
B Quotients

Can we impose additional relations on TL?

If g is a relation, then

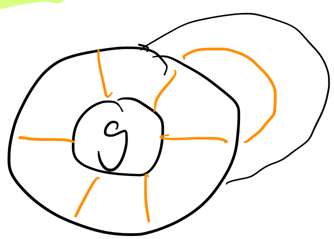


$= 0 \Rightarrow$



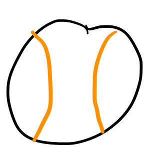
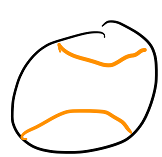
$= 0 \quad \forall h$

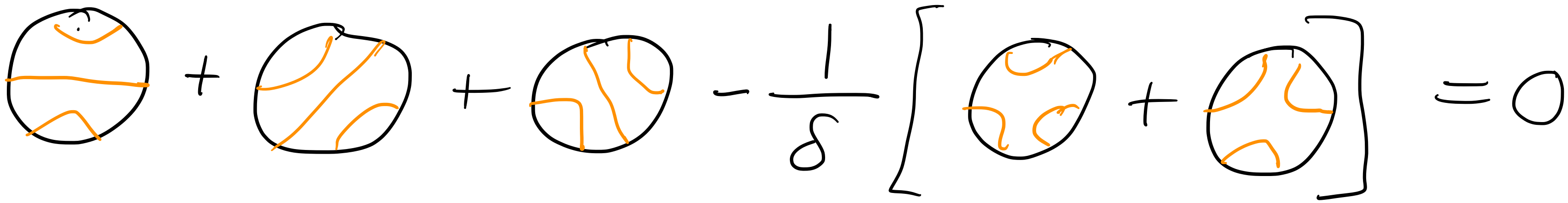
g minimal $\Rightarrow$



$= 0 \quad \forall$ caps

First two solutions to these cap eq'ns:

$g =$  $+ \, \delta \cdot$  , $\delta^2 - 1 = 0$ ($\mathbb{Z}/2$ homology)



$\delta^2 - 2 = 0$

$\rightsquigarrow$ "Ising TQFT", much studied in quantum computation

In general, $\exists$ sol'n to cap equations whenever $\delta = q + q^{-1}$, $q^r = -1$

$\rightsquigarrow$ <u>full</u> 2+1-dim'l TQFT (not just $(2+\varepsilon)$-dim'l)

(Turaev-Viro TQFT)

Goal: Generalize all of the above (including (A) + (B)) to higher dimensions

TL: codim 1 submanifolds of B^2

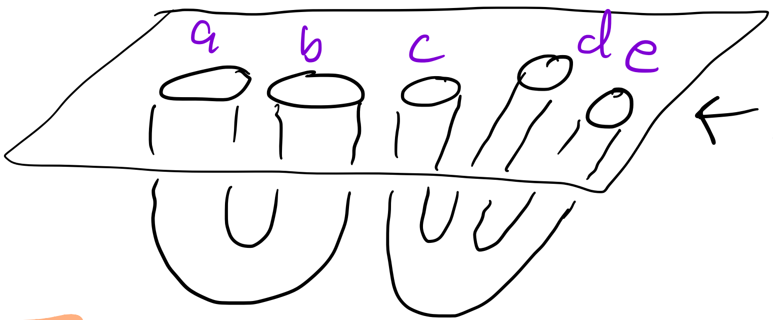
HTL: codim k submanifolds of B^n

What are the local relations?

Given $S \subset B^n$, get partition of $\pi_0(\partial S)$

(proper embedding of codim k submanifold)

$a, b \in \pi_0(\partial S)$ in same block $\Leftrightarrow$ a, b are in same component of S:



$\rightsquigarrow [ab|cde]$

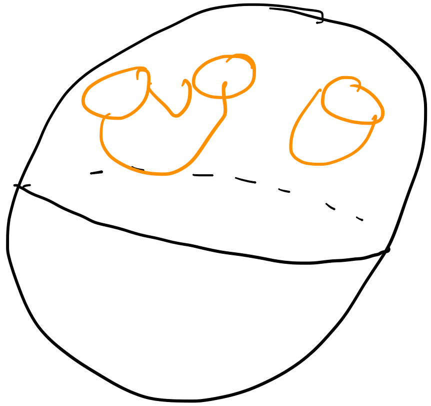
Define $A(B^n; c) = \mathbb{C}\left[\{S \subset B^n ; \partial S = c\}\right] / \sim$

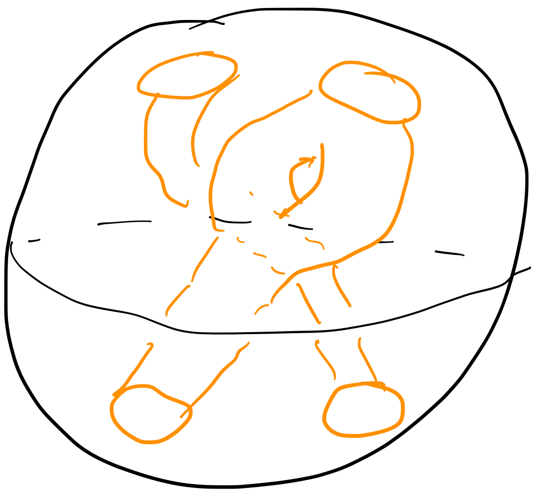
(c: codim K submanifold of ∂B^n)

$S \sim S'$ if S and S' induce same partition of c

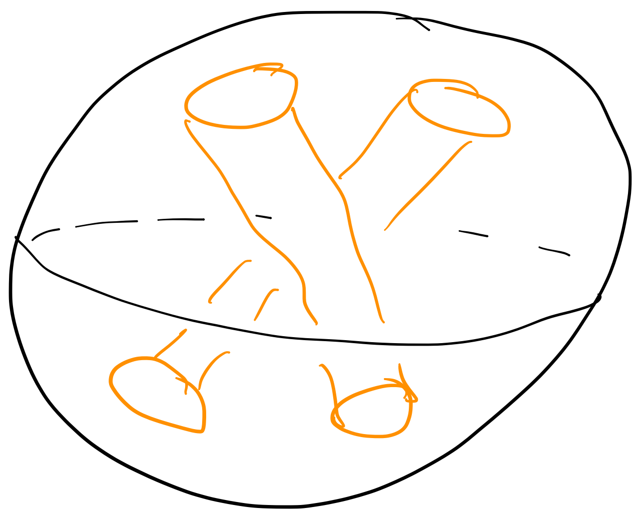
also, closed components $\rightsquigarrow$ factor of $\beta \in \mathbb{C}$

$= \beta \circ$





$=$



Everything behaves well w.r.t. gluing

$\Rightarrow$ $(n+\varepsilon)$-dim'l TQFT

- $A(M^n; c) = \mathbb{C}\left[\{S \subset M, \partial S = c\}\right] / \sim$

predual Hilbert space

codim k subman

$S \sim S'$ if they are related by local partition-preserving modification

- $A(Y^{n-1}; c) = \left[\begin{array}{l} \text{obj: } \{\text{codim } k \; S \subset Y\} \\ \text{mor}(a \to b) = A(Y \times I; a, b) \end{array}\right.$

- Path integral $Z(W^{n+1}) : A(\partial W) \to \mathbb{C}$, defined when $W = $ 0-handles $\cup$ 1-handles

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$n=2$, $k=1$ $\rightsquigarrow$ Temperley-Lieb

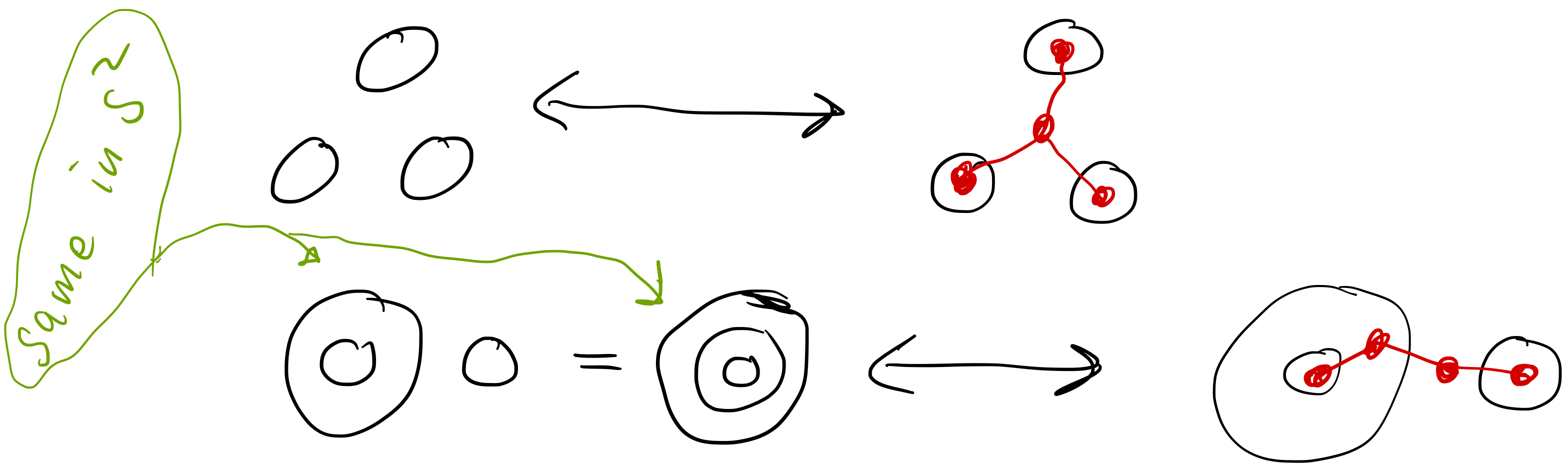
$n=3$, $k=1$ $\rightsquigarrow$ rest of this talk

Step 1. Understand $A(B^n; c)$

$$C \in \{\text{circles in } \partial B^3\} \longleftrightarrow \{\text{unrooted trees}\}$$

up to isotopy

dual graph



Define $D(T) = \dim(A(B^3; T))$

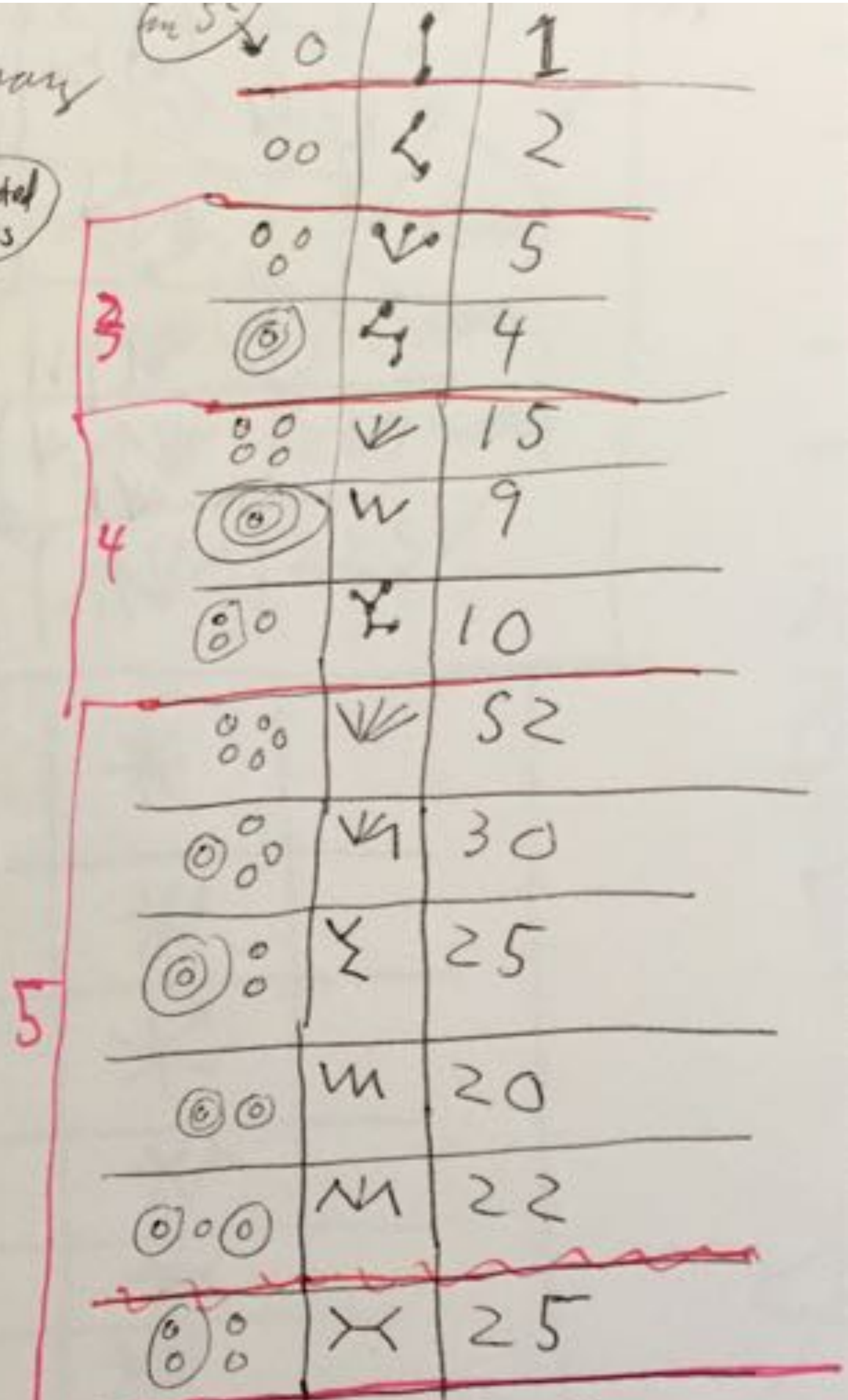
$=$ # of admissible partitions of the edges of T

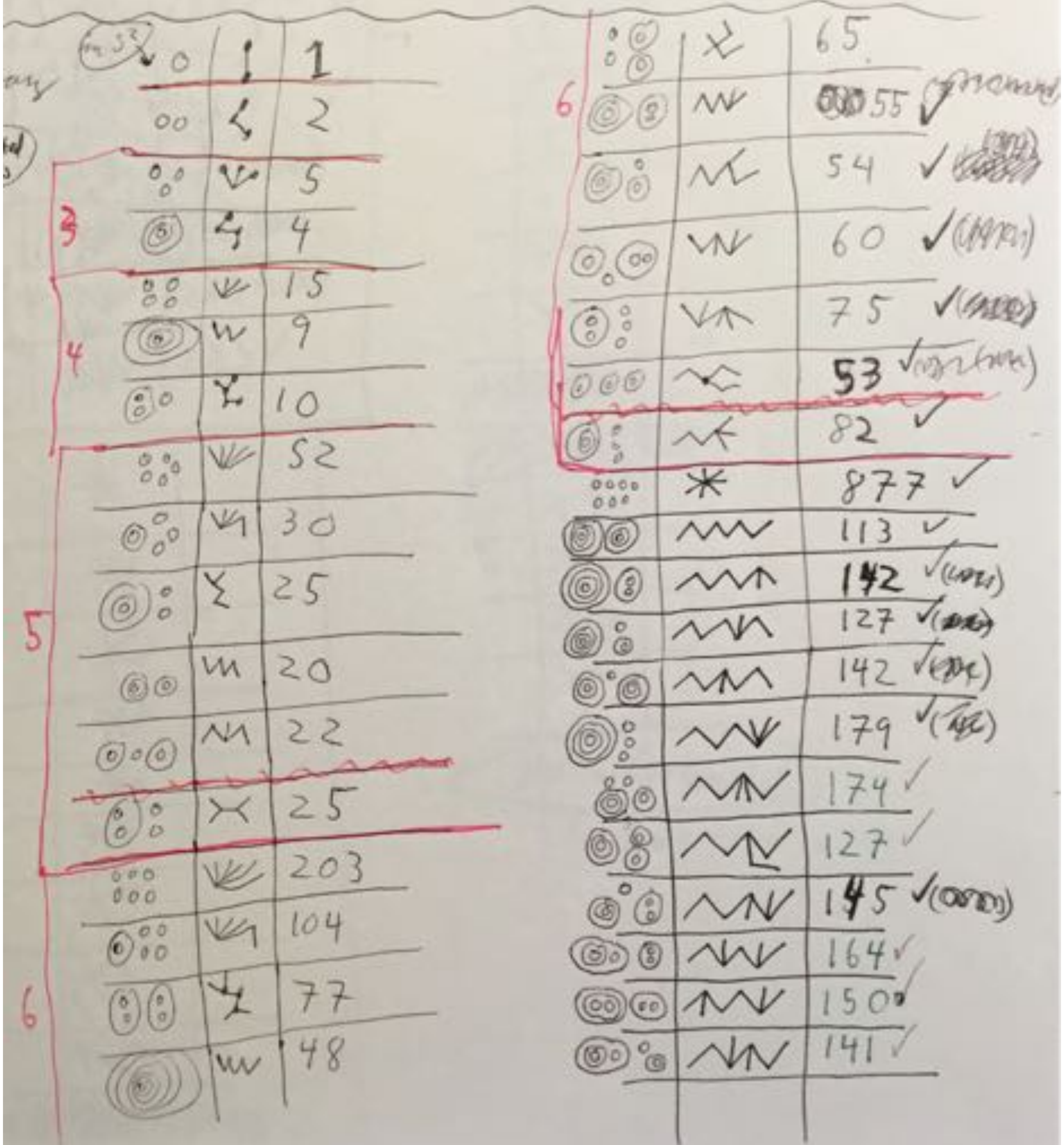
3-dim'l Catalan numbers

Summary

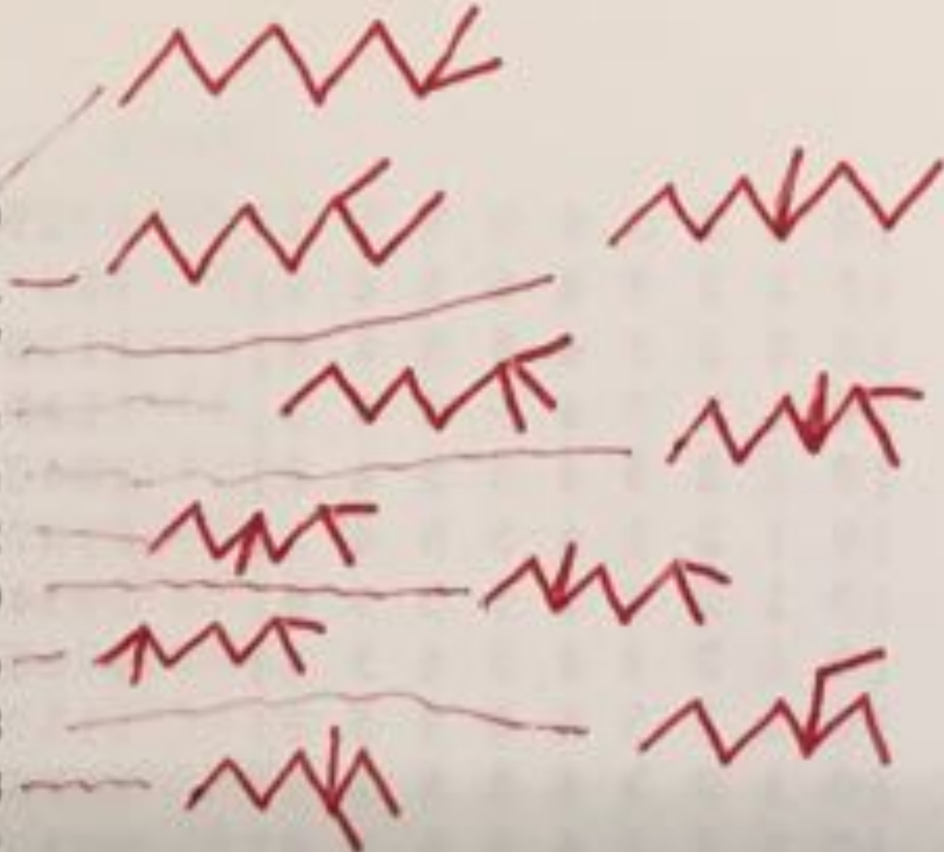
unrooted trees

1
2
3
6
11
23
47
106

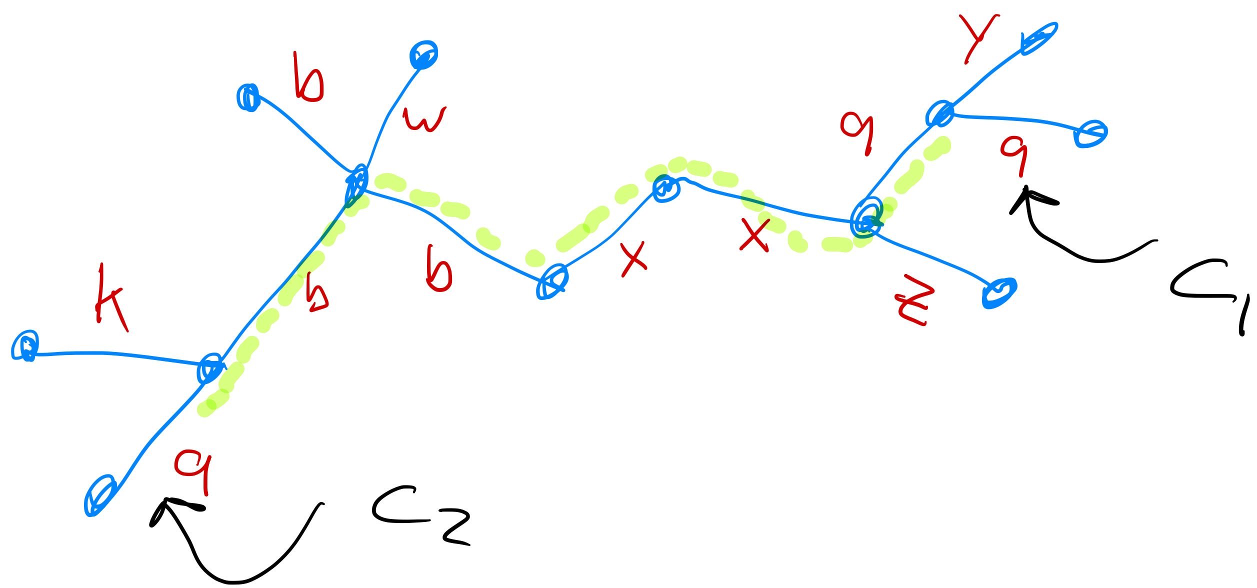



1
2
3
5
4
4
15
9
10
5
52
30
25
20
22
25
6
203
104
77
48
65
6
55
54
60
75
53
82
877
113
142
127
142
179
174
127
145
164
150
141

[0 1 2 3 4 5 6 6]	329
[0 1 2 3 4 5 6 5]	321
[0 1 2 3 4 5 6 4]	327
[0 1 2 3 4 5 5 5]	494
[0 1 2 3 4 5 5 4]	364
[0 1 2 3 4 5 5 3]	462
[0 1 2 3 4 5 5 2]	369
[0 1 2 3 4 5 5 1]	455
[0 1 2 3 4 5 4 7]	318
[0 1 2 3 4 5 4 4]	403
[0 1 2 3 4 5 4 3]	362
[0 1 2 3 4 5 4 2]	371
[0 1 2 3 4 5 3 7]	325
[0 1 2 3 4 5 3 3]	496
[0 1 2 3 4 4 4 4]	666
[0 1 2 3 4 4 4 3]	463
[0 1 2 3 4 4 4 2]	586

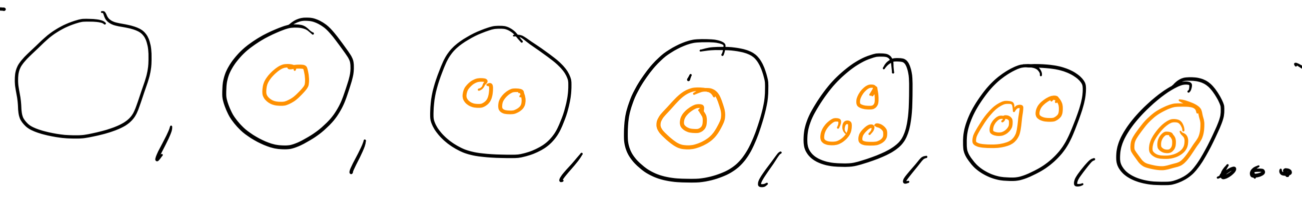


Prop. A partition P of the edges of a tree T is admissible $\iff$ $\forall$ edges c_1, c_2 in same block a of P, $\forall$ blocks $b \neq a$, the path from c_1 to c_2 crosses edges of b an <u>even</u> number of times.



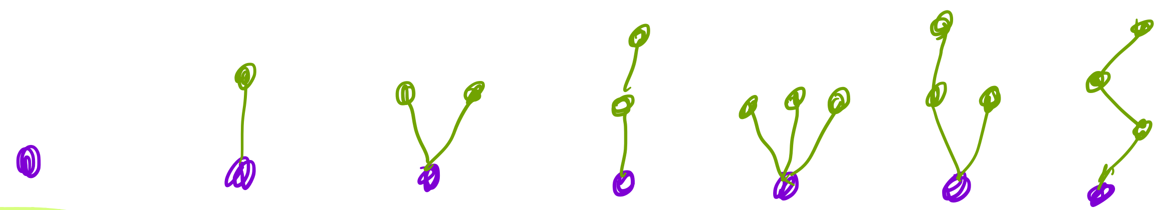
Step 2. Understand idempotents of $A(D^2; c)$, $c = 0$ or 2 or 4 or 6 .... pts in ∂D^2

$c=0$ case

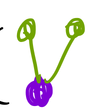
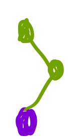
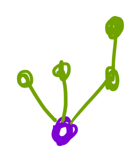
$\text{obj}(A(D^2; \emptyset)) = \{$  $\}$

$\longleftrightarrow$ rooted trees

(dual graph)



$$\dim(\text{mor}(T_1 \to T_2)) = D(T_1 \vee T_2)$$

e.g. $\dim($mor( $\to$  $)) = D($  $) = D($   $) = 10$

Let $\{e_i\}$ be a complete set of minimal idempotents for $A(D^2;\emptyset)_0$. Then

$$T \cong \bigoplus_i \langle e_i, T\rangle \cdot e_i$$

for any object T of $A(D^2;\emptyset)$, where

$$\langle e_i, T\rangle = \mathrm{mor}(e_i \to T).$$

It follows that $\mathrm{mor}(T_1 \to T_2) \cong \bigoplus_i \langle e_i, T_1\rangle \otimes \langle e_i, T_2\rangle$

hence

$$D(T_1 \vee T_2) = \dim(\mathrm{mor}(T_1 \to T_2)) = \sum_i \dim\langle e_i, T_1\rangle \cdot \dim\langle e_i, T_2\rangle$$

$\forall T_1, T_2$

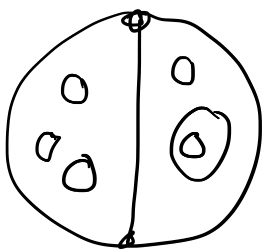
(!) →

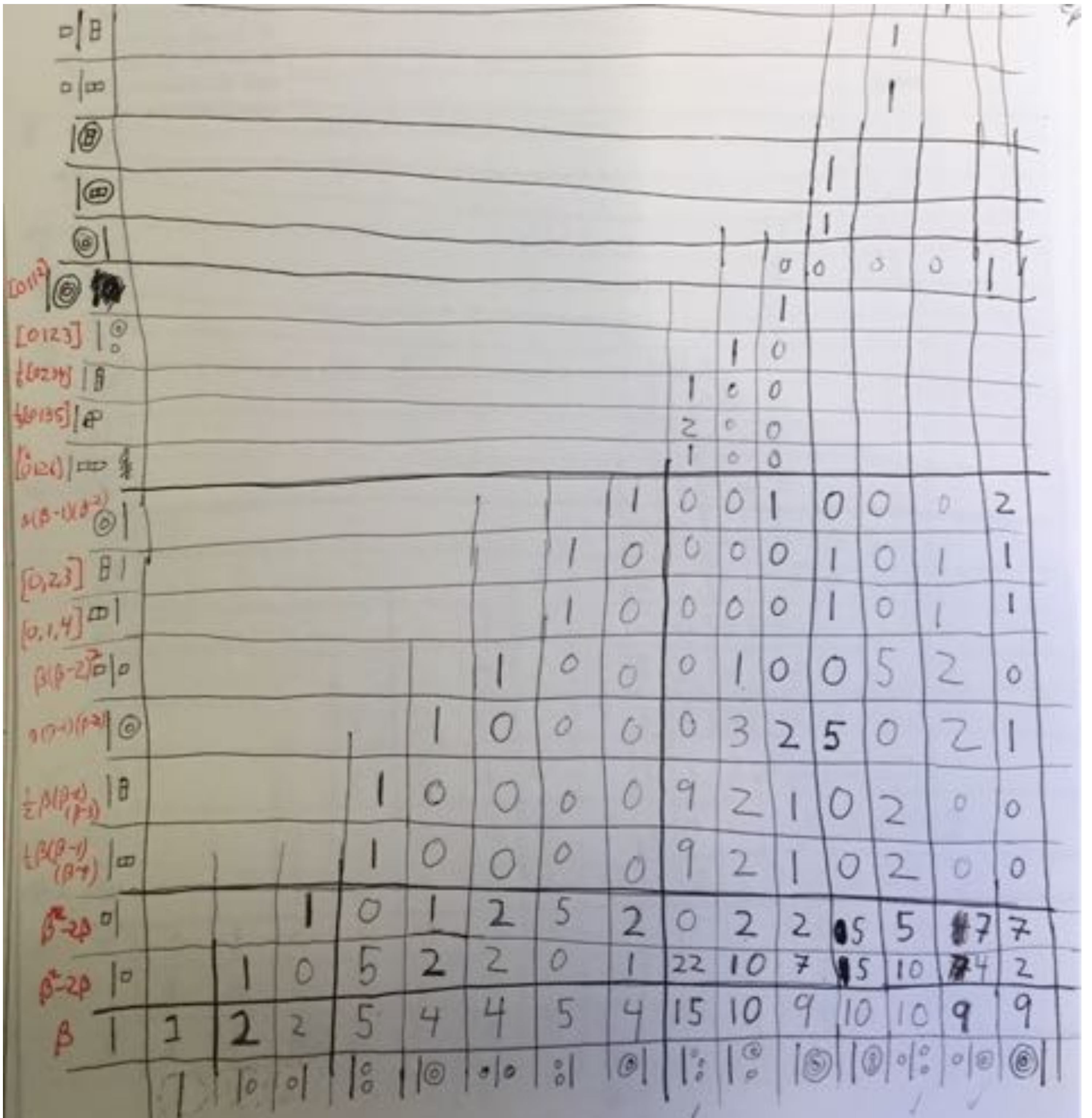
												1					
											1						
											1						
										1	0						
										1	0						
									1	0	0						
$\frac{1}{2}\beta(\beta-2)(\beta-3)$								1	1	1	0	0	2	2	0	9	0
$\frac{1}{2}\beta(\beta-1)(\beta-4)$								1	1	1	0	0	2	2	0	9	0
$\beta(\beta-2)^2$							1	0	1	4	5	2	1	5	0	0	0
$\beta(\beta-1)(\beta-2)$						1	0	0	2	0	0	2	3	0	5	0	0
$\frac{1}{6}(\beta-1)(\beta-2)(\beta-3)$					1	0	0	0	0	0	2	1	0	0	1	0	10
$\frac{1}{3}\beta(\beta-2)(\beta-4)$					2	0	0	0	0	0	4	2	0	0	2	0	20
$\frac{1}{6}\beta(\beta-1)(\beta-6)$					1	0	0	0	0	0	2	1	0	0	1	0	10
$\beta^2-2\beta$				1	0	2	2	5	8	9	5	4	10	10	5	22	0
$\frac{1}{2}(\beta^2-3\beta+2)$			1	0	6	1	2	0	2	4	12	8	2	5	6	0	31
$\frac{1}{2}[\beta^2-3\beta]$			1	0	6	1	2	0	2	4	12	8	2	5	6	0	31
$\beta-1$		1	3	2	10	5	6	5	11	13	20	16	12	15	15	15	37
1	1	1	2	2	5	4	4	5	9	9	10	9	10	10	10	15	15

√
282

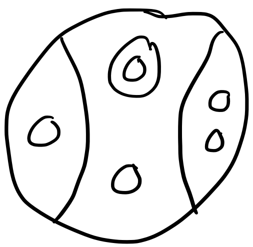
Similar analysis holds for

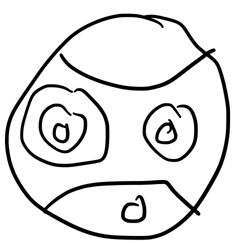
$A(D^2; 2)$

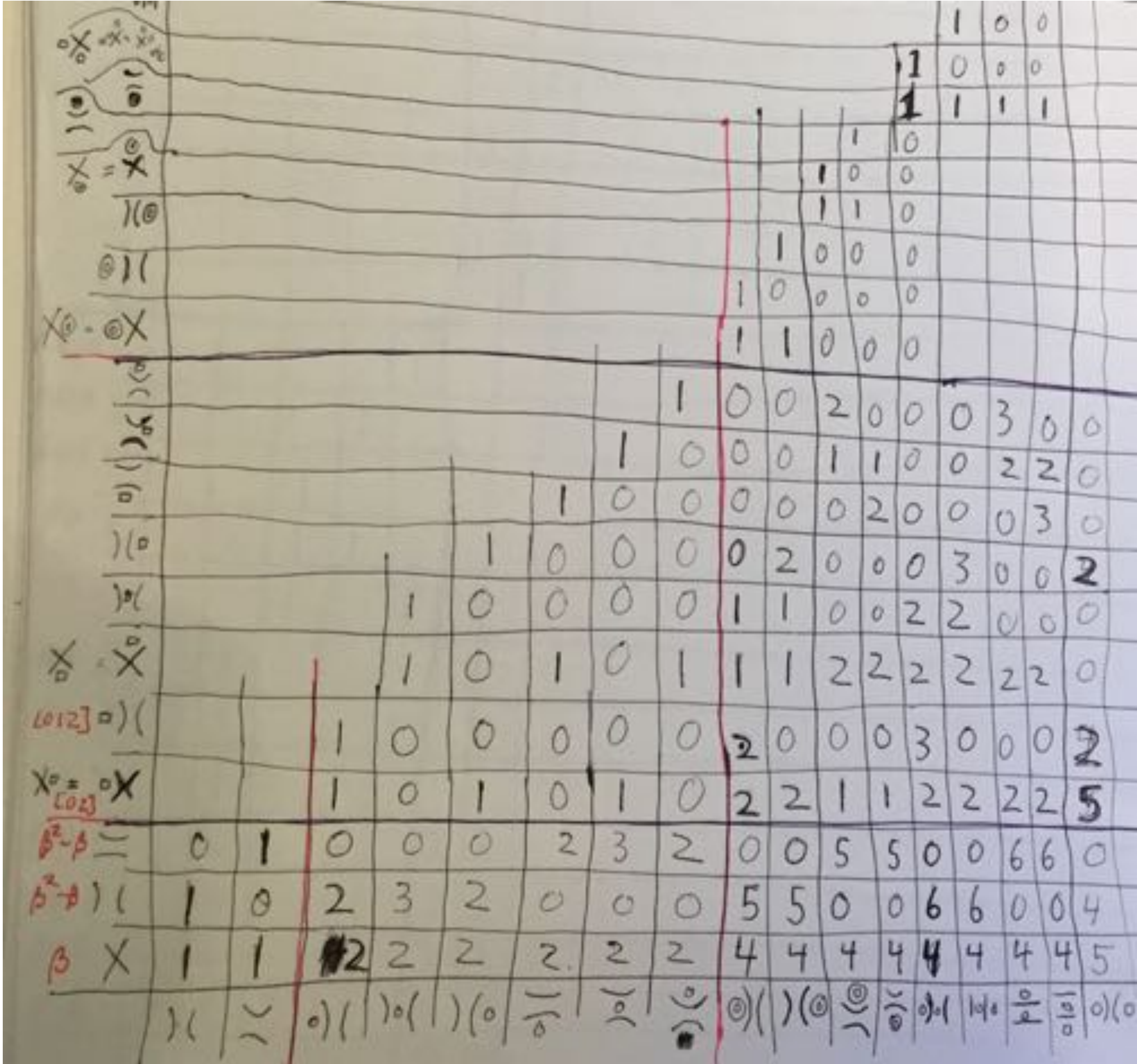




and for $A(D^2; 4)$







General case

- Let T be a rooted tree and let $Sym(T)$ be its symmetry group. Note that $Sym(T)$ embeds in $mor(T \to T)$ via cylinders

- Let ρ be an irrep of $Sym(T)$ and let f_ρ be a corresponding minimal idempotent of the group algebra of $Sym(T)$.

- Then $\exists$ a minimal idempotent of $A(D^2; \phi)$ of the form

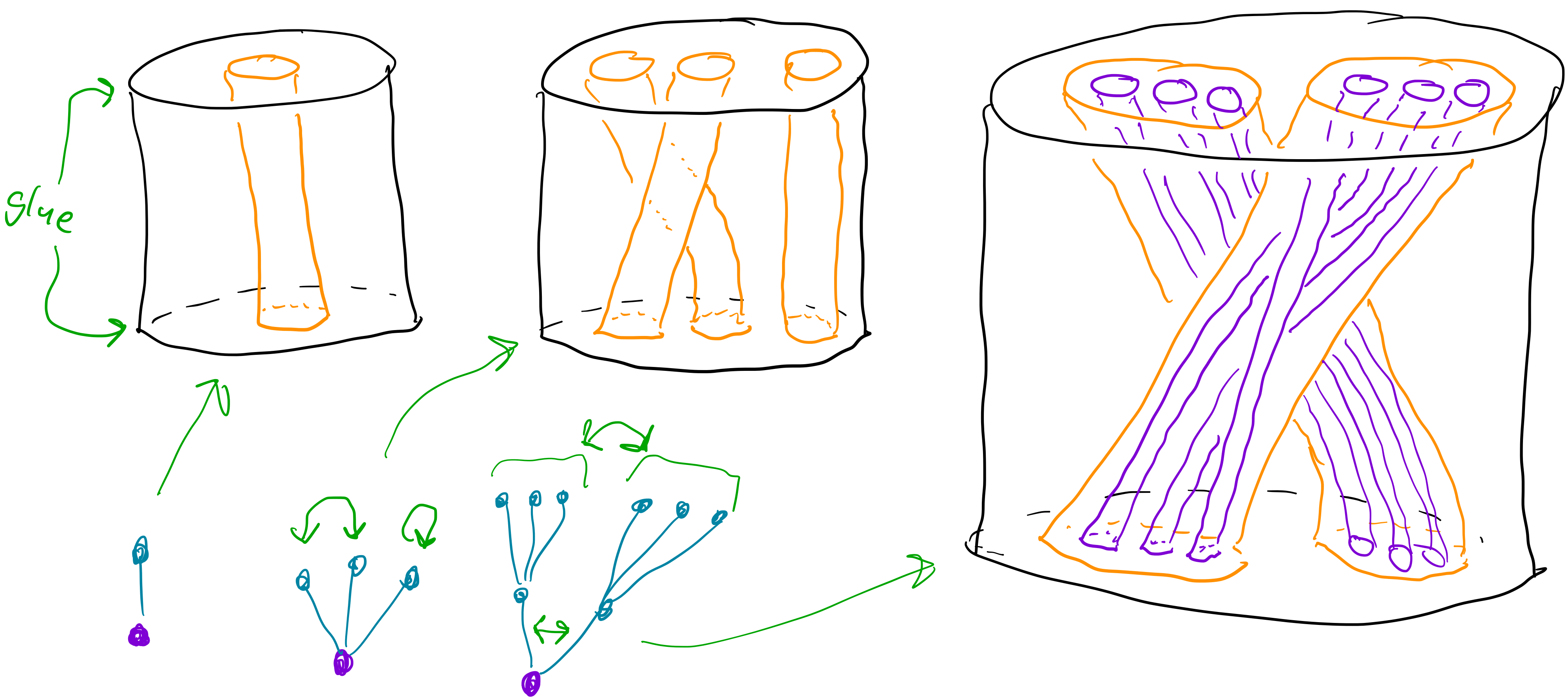
$$e_\rho = f_\rho + \langle \text{lower order terms} \rangle$$

Thm. (Classification of minimal idempotents of $A(D^2;\emptyset)$): $\{e_\rho\}$ from above is a complete set of minimal idempotents of $A(D;\emptyset)$.

Note: $\exists$ similar result for $A(D^2;2k)$

Step 3. Polynomial generators of $mor(\emptyset \to \emptyset) \subset A(S^1 \times I)$

Sample elements of $mor(\emptyset \to \emptyset)$:



- In general, let T be a rooted tree and let $\alpha \in \mathrm{Sym}(T)$.
- $cl(\alpha) \in \mathrm{mor}(\emptyset \to \emptyset) \subset A(S^1 \times I; \emptyset)$

  ↑ closure / mapping cylinder
- $cl(\alpha) = cl(\alpha')$ iff α and α' are conjugate in $\mathrm{Sym}(T)$.
- $cl(\alpha) \cdot cl(\beta) = cl(\alpha \vee \beta)$

  ↑ composition in $\mathrm{mor}(\emptyset \to \emptyset)$; ↑ wedge of trees
- $cl(\alpha)$ is primitive $\Longleftrightarrow$ α has a single orbit when acting on edges adjacent to the root of T.

Thm. (a) $\{cl([\alpha])\}$ is a basis of $mor(\emptyset \to \emptyset)$ ← conj class

(b) $mor(\emptyset \to \emptyset)$ is a commutative polynomial algebra with generators $\{cl([\beta]) \mid \beta \text{ primitive}\}$

Step 4. TQFT trace/I.P./measure

Thm $\{cl(e_p)\}$ is an orthonormal basis of $mor(\emptyset \to \emptyset)$ w.r.t. TQFT inner product

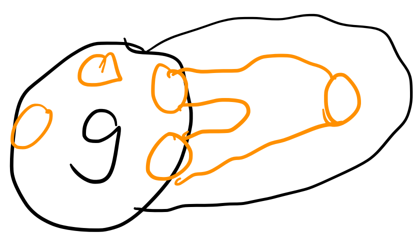
Cor. $\exists$ family of orthogonal polynomials

- variables indexed by pairs $(T, [\alpha])$
  - rooted tree (T)
  - conj class of primitive symmetry of T ($[\alpha]$)
- polynomials indexed by pairs (T, ρ)
  - rooted tree (T)
  - irrep of $Sym(T)$ (ρ)

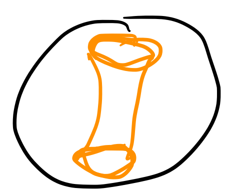
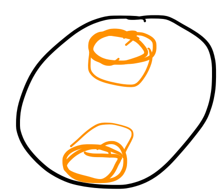
B Quotients

Let $g \in A(B^3; c)$. g is a minimal $\otimes$-ideal generator $\iff$

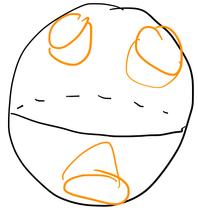
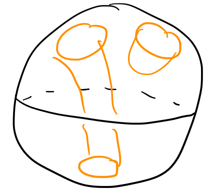
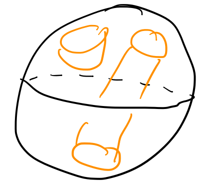
$= 0 \quad \forall$ caps



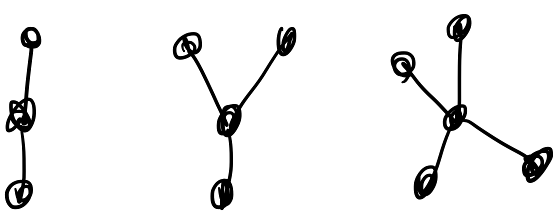
$= 0 \quad \forall$ joining saddles

Example: $\beta = I$, $g =$  $-$ 

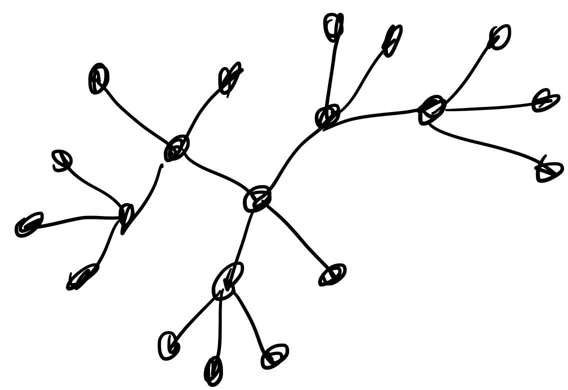
Example:

$\beta = 2$, $g =$  $-$ $\Big[$  $+$  $+$  $\Big]$ $+ 2 \cdot$ 

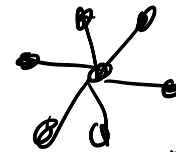
Thm. A minimal ideal generator $g \in \mathcal{A}(B^3; c)$ exists iff $\beta = k \in \mathbb{N}_+$ and the ∂-condition c is a tree of constant non-twig valence $k+1$.



$\beta = 1$ $\quad$ $\beta = 2$ $\quad$ $\beta = 3$



$\beta = 3$

If c is a star (), then coefficient of a partition with block sizes $a_1, \dots, a_m$ is

$$\prod_i (-1)^{a_i - 1} (a_i - 1)!$$